18/02/2026

Some important definitions

Monotonic decreasing sequence / Nested sequence

Let $\{S_n\}$ be a sequence of subsets of X in a metric space (X, d).

If $S_1 \supset S_2 \supset S_3 \supset \cdots$ then this sequence $\{S_n\}$ is called a decreasing sequence or monotonic sequence.

CANTOR'S INTERSECTION THEOREM

Statement: Let X be a complete metric space. Let $\{F_n\}$ be a decreasing sequence of non-empty closed subsets of X such that $\lim_{n\to\infty} d(F_n) = 0$ (i.e. whose diameters tend to zero, i.e. $d(F_n) \to 0$). Then $F = \bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

Proof: Given that (X, d) is a complete metric space. Also $\{F_n\}$ is a decreasing sequence of non-empty closed subsets of X such that

$$d(F_n) \to 0.$$

We have to prove that $\bigcap_{n=1}^{\infty} F_n \neq \phi$

and contains exactly one point. $\frac{2}{4}$

Now $\lim_{n \to \infty} d(F_n) = 0$

$\Rightarrow$ $\sup \{ d(x,y) : x, y \in F_n \} \to 0$

$\Rightarrow$ $\bigcap_{n=1}^{\infty} F_n$ can not contain more than one point.

so it is sufficient to prove that

$\bigcap_{n=1}^{\infty} F_n$ is non-empty.

Given that each F_i is non-empty. Hence we can have a sequence $\{x_n\}$ such that

$x_n \in F_n$, $n = 1, 2, 3, \ldots$

Ist we prove that $\{x_n\}$ is a cauchy sequence.

$\because$ $d(F_n) \to 0$ $\Rightarrow$ For every $\epsilon > 0$ $\exists$ a (+)ve integer m_0 such that $d(F_{m_0}) < \epsilon$.

$\because$ $\{F_n\}$ is a decreasing sequence

$\Rightarrow$ $F_n, F_m \subset F_{m_0}$ $\forall n, m \geq m_0$

$\Rightarrow$ $x_n, x_m \in F_{m_0}$

$\Rightarrow$ $d(x_n, x_m) < \epsilon$ (by definition of diameter :)

$\Rightarrow \{x_n\}$ is a Cauchy sequence.

Since (X, d) is a complete metric space, it will converge to a point $x_0 \in X$.

Now, we show that $x_0 \in \bigcap_{n=1}^{\infty} F_n$.

Let $x_0 \notin \bigcap F_n$

$\Rightarrow x_0 \notin F_{n_0}$ for some $n_0 \in N$.

Since each F_n is closed $\Rightarrow F_{n_0}$ is also closed.

$\Rightarrow x_0$ cannot be an accumulation point of F_{n_0}

$\Rightarrow d(x_0, F_{n_0}) \neq 0$

Let $d(x_0, F_{n_0}) = \delta > 0$

By definition of $d(x, A) = \inf \{d(x, y) : y \in A\}$.

$\Rightarrow d(x_0, y) \geq \delta \ \forall \ y \in F_{n_0}$

$\Rightarrow F_{n_0}$ and open sphere $S(x_0, \frac{\delta}{2})$ are disjoint i.e. $F_{n_0} \cap S(x_0, \frac{\delta}{2}) = \phi$

Now $n > n_0 \Rightarrow F_n \subset F_{n_0}$

$\because x_n \in F_n \Rightarrow x_n \in F_{n_0}$ But $F_{n_0} \cap S(x_0, \frac{\delta}{2}) = \phi$

$\Rightarrow x_n \notin S(x_0, \frac{\delta}{2})$ which is not possible.

$\Rightarrow$ our assumption is wrong.

our assumption that

$$x_0 \notin \bigcap_{n=1}^{\infty} F_n$$ is wrong.

$$\Rightarrow x_0 \in \bigcap_{n=1}^{\infty} F_n.$$

Hence the theorem